Power Monoids in a New Framework for Factorization

joint work with S. Tringali

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Factorization theory

FACTORIZATION THEORY studies existence and (non-)uniqueness of factorizations into *"indecomposable"* blocks of elements in rings and monoids.

In $\mathbb{Z}$: $120 = 2^3 \cdot 3 \cdot 5$	Fundamental Theorem of arithmetic
In $\mathbb{Z}[X]$: $x^3 - 1 = (x-1)(x^2+x+1)$	Unique factorization
In $\mathbb{Z}\left[\sqrt{-5}\right]$: $6 = 2 \cdot 3 = (1+\sqrt{-5})(1-\sqrt{-5})$	Same length, but different
In $\mathbb{Q}\left[x^2, x^3\right]$: $x^6 = (x^2)^3 = (x^3)^2$	Finitely many different lengths
In $\langle \frac{1}{p} \mid p \text{ prime} \rangle \subseteq (\mathbb{Q}, +)$: $1 = \underbrace{\frac{1}{p} + \ldots + \frac{1}{p}}_{p \text{ times}}$	Infinitely many different lengths

Factorization problems

The study of factorization can be outlined in the following steps:

- Consider a class of objects $\mathcal{C}$;
- Let $\mathcal{A}(\mathcal{C})$ be a set of building blocks for this class;
- Check if (or under which conditions) every element in $\mathcal{C}$ can be decomposed into elements of $\mathcal{A}(\mathcal{C})$ (EXISTENCE);
- Study the (NON-)UNIQUENESS (however defined) of such factorizations.

EXAMPLE: Let $\mathcal{C} = \mathbb{N}_{\geq 2}$ and $\mathcal{A}(\mathcal{C})$ be the set of primes.
Every element of $\mathcal{C}$ factors as a product of elements of $\mathcal{A}(\mathcal{C})$ and this decomposotion is *unique* (up to the order of factors).

Some example of factorizations

...from different areas of mathematics

- Factorization of non-units into products of irreducibles in multiplicative monoids (rings, domains)
- Factorization of permutations as composition of transpositions
- Factorization of singular matrices over integral domains as product of idempotent matrices
- Decomposition of objects in categories
- Direct sum decomposition of modules
- Factorization of ideals into prime, primary, radical, ... ideals
- . . .

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The “classical” factorization theory

FOCUS OF THE CLASSICAL THEORY OF FACTORIZATION:

Factorizations into ATOMS (non-units that cannot be written as a product of two non-units) in COMMUTATIVE and/or CANCELLATIVE ($ax = bx$ or $xa = xb$ implies $a = b$) monoids.

Research question

Can we build a general theoretical framework in which we can study any factorization problem, independently from the nature of the building blocks and the properties of the monoid in which the factorization takes place?

Roughly speaking...

What if we want to study factorizations into building blocks that are not atoms, in monoids that may be neither cancellative nor commutative?

WE CAN DO IT,

in the context of...

... a new Theory of Factorization (C. & Tringali, since 2021)

GOAL OF THE THEORY: Given a monoid M and two *arbitrary* subsets $S, A \subseteq M$, study the EXISTENCE and NON-UNIQUENESS of decompositions of the elements of S as products of elements of A.

STRATEGY: Combine the languages of monoids and preorders (*reflexive* and *transitive* binary relations on a set).

(I) WORK WITH "ARBITRARY" FACTORS

$(M, \preceq) \longrightarrow$ general notion of "IRREDUCIBLE" element of M.

For suitable choices of $\preceq$, the "irreducibles" of M, i.e., the building blocks of the factorizations, can be even invertible or idempotent elements of M.

A general existence result

Theorem (Tringali 2021, C. & Tringali 2024 a)

Let M be a monoid and $\preceq$ a preorder on M. If $\preceq$ is artinian, for every $d \geq 2$, every $\preceq$-non-unit of M (generalization of the usual notion of non-unit) factors into a product of $\preceq$-irreducibles of degree d (generalization of atoms).

L. C., S. TRINGALI, Israel J. Math, 2024.

Taking $\preceq$ as the DIVISIBILITY PREORDER the theorem generalizes classical existence results for *atomic* factorization.

For *suitable* choices of $\preceq$ and M, (among other results) we obtain as a corollary of the above theorem results concerning factorizations into idempotents, units, ecc.

(II) Introduce and use minimal factorizations

$$(\mathscr{F}(M), \sqsubseteq) \longrightarrow \text{MINIMAL FACTORIZATIONS}.$$

Minimal factorizations are a refinement of classical factorizations that counter the blow-up phenomena of the arithmetical invariants which are typical of non-atomic factorizations.

For example, for the case of idempotent factorizations, minimal factorizations do not take into account repeated factors.

Given a pair $(M, \preceq)$:

- we found sufficient conditions for the (minimal) factorizations in M to be bounded in length or finite in number (if measured or counted in a suitable way).

  L. C., S. Tringali, J. Algebra 630, 2023.

- we introduced arithmetical invariants (sets of minimal lengths, unions of sets of minimal lengths, minimal elasticity) associated to minimal factorizations and found interesting finiteness results for them.

  L. C., S. Tringali, Ark. Math, 2024.

In this talk we explore the arithmetic of POWER MONOIDS
in the framework of this new theory of factorization.

The preorder we are going to implicitly consider in our discussion is the
DIVISIBILITY PREORDER

S. TRINGALI, J. Algebra 602, 2022.

L. C., S. TRINGALI, J. Algebra 630, 2023.

L. C., S. TRINGALI, J. Algebra, in press.

What is a power monoid?

H = (multiplicative) monoid with identity 1_H.
$H^\times$ = group of units or invertible elements of H.

The set

$$\mathcal{P}(H) := \{X \subseteq H \colon X \neq \emptyset\}$$

is a monoid with respect to **setwise multiplication**

$$(X, Y) \mapsto XY := \{xy \colon x \in X,\ y \in Y\},$$

called LARGE POWER MONOID of H.

The submonoids of $\mathcal{P}(H)$ will be generically called POWER MONOIDS (PMS), and H will be referred to as their GROUND MONOID.

Why are PMs interesting?

- PMs are key object of study in semigroup theory since 1960s.
  - **Tamura and Shafer's isomorphism problem (1967):** *is it true that if $\mathcal{P}(H)$ is isomorphic to $\mathcal{P}(K)$, then H is isomorphic to K?*

    T. Tamura, J. Shafer, Math. Japon. 12, 1967.

    Answered (in the negative) by Mogiljanskaja (1973) with a counterexample in the infinite case, the question is still open for finite monoids.
- PMs are a natural algebraic framework for famous problems in additive NT:
  - **Sarkozy's conjecture:** For all but finitely many primes p, the set $\mathcal{R}_p \subseteq \mathbb{F}_p$ of quadratic residues $\bmod p$ is an atom[1] in $\mathcal{P}(H)$, with $H = (\mathbb{F}_p, +)$.

    A. Sárközy, Acta Arith. 115, 2012.
  - **Inverse Goldbach conjecture:** Every set of integers that differ from the set of (positive rational) primes by finitely many elements is an atom in $\mathcal{P}(H)$, with $H = (\mathbb{Z}, +)$.

    C. Elsholtz, Mathematika 48, 2001.
- PMs have long been known to play a central role in the theory of automata and formal languages.

  J. Almeida, Semigroup Forum 64, 2002.

[1] An atom is a non-unit which is not the sum of two non-units.

Why are PMs interesting?

- PMs have a rich and interesting arithmetics, as fristly observed by Fan and Tringali in 2018.

  Y. FAN, S. TRINGALI, J. Algebra 512, 2018.

  Since then, they were further considered from this perspective in:

  A. ANTONIOU, S. TRINGALI, Pacific J. Math 312, 2021.

  P. Y. BIENVENU, A. GEROLDINGER, Israel J. Math., 2024.

  A. REINHART, arXiv:2508.10209, 2025.

  They have also been the subject of a CrowdMath project launched by F. Gotti in 2023 within the MIT Primes program:

  `https://artofproblemsolving.com/polymath/mitprimes2023`

  A. AGGARWAL, F. GOTTI & S. LU, arXiv:2412.05857, 2024.

  J. DANI, F. GOTTI, L. HONG, B. LI & S. SCHLESSINGER, arXiv:2501.03407, 2025.

- Due to their “high non-cancellativity”, PMs are a leading example in the *new unifying theory of factorization* developed by C. and Tringali.

Power monoids are wild objects

The large power monoid $\mathcal{P}(H)$ of H is a rather complicated object. It is often useful to focus on "FINITARY"(OR "SMALL") POWER MONOIDS:

$\mathcal{P}_{\text{fin}}(H) := \{X \in \mathcal{P}(H) \colon |X| < \infty\}$, the SMALL PM of H;

$\mathcal{P}_{\text{fin},\times}(H) := \{X \in \mathcal{P}_{\text{fin}}(H) \colon X \cap H^\times \neq \emptyset\}$, the RESTRICTED (SMALL) PM of H;

$\mathcal{P}_{\text{fin},1}(H) := \{X \in \mathcal{P}_{\text{fin}}(H) \colon 1_H \in X\}$, the REDUCED (SMALL) PM of H.

If H is DEDEKIND-FINITE, i.e., if $xy = 1_H$ implies $yx = 1_H$, then:

- $\mathcal{P}_{\text{fin},1}(H)$ and $\mathcal{P}_{\text{fin},\times}(H)$ have the same length sets (for factorizations into irreducibles);
- $\mathcal{P}_{\text{fin},\times}(H)$ is a divisor-closed submonoid of $\mathcal{P}_{\text{fin}}(H)$.

If H is CANCELLATIVE, then $\mathcal{P}_{\text{fin}}(H)$ is divisor-closed in $\mathcal{P}(H)$.

At least for H Dedekind finite, there is much about PMs that we can understand from the investigation of $\mathcal{P}_{\text{fin},1}(H)$.

Recent works on reduced power monoids

- Bienvenu and Geroldinger have recently addressed ideal-theoretic and analytic properties of $\mathcal{P}_{\mathrm{fin},0}(S)$, where S is a **numerical semigroup**[2].

Conjecture: For S and S' numerical monoids, $\mathcal{P}_{\mathrm{fin},0}(S) \cong \mathcal{P}_{\mathrm{fin},0}(S') \Leftrightarrow S = S'$.

P. Y. Bienvenu, A. Geroldinger, Israel J. Math., 2024.

- The conjecture was settled in positive by Tringali and Yan, who proved a more general result:

For S and S' Puiseux monoids[3], $\mathcal{P}_{\mathrm{fin},0}(S) \cong \mathcal{P}_{\mathrm{fin},0}(S') \Leftrightarrow S \cong S'$.

S. Tringali & W. Yan, Proc. Amer. Math. Soc., 2025.

- More results on (or related to) the isomorphism problem have followed

S. Tringali & W. Yan, J. Combin. Theory Ser. A, 2025.

B. Rago, Proc. Amer. Math. Soc., to appear.

S. Tringali & W. Yan, preprint (available upon request), 2025.

[2] A **numerical semigroup** is a submonoid S of $(\mathbb{N}, +)$ such that $\mathbb{N} \setminus S$ is finite.

[3] A **Puiseux monoid** is a submonoid of $(\mathbb{Q}_{\geq 0}, +)$.

Recent works on reduced power monoids

- The **arithmetic** of $\mathcal{P}_{\text{fin},1}(H)$ is the main object of study in

Y. FAN, S. TRINGALI, J. Algebra 512, 2018.

A. ANTONIOU, S. TRINGALI, Pacific J. Math 312, 2021.

A. AGGARWAL, F. GOTTI & S. LU, arXiv:2412.05857, 2024.

A. REINHART, arXiv:2508.10209, 2025.

In our work we added to this line of research. In this talk, we are going to:

- define minimal factorizations into *irreducibles* of non-unit elements of reduced power monoids;
- characterize reduced power monoids for which such minimal factorizations are *unique*.

Basics on $\mathcal{P}_{\mathrm{fin},1}(H)$

Let H be a monoid and let $X, Y \in \mathcal{P}_{\mathrm{fin},1}(H)$. The following hold:

- If X is a divisor of Y in $\mathcal{P}_{\mathrm{fin},1}(H)$, then $X \subseteq Y$.

  If $Y = UXV$ for some $U, V \in \mathcal{P}_{\mathrm{fin},1}(H)$, then $X = \{1_H\}X\{1_H\} \subseteq UXV = Y$.

- X and Y are associated in $\mathcal{P}_{\mathrm{fin},1}(H)$ if and only if $X = Y$.

  Since X and Y are associated in $\mathcal{P}_{\mathrm{fin},1}(H)$ if and only if $X \mid Y$ and $Y \mid X$.

- $\mathcal{P}_{\mathrm{fin},1}(H)$ is a reduced[4], Dedekind-finite monoid.

  If $XY = \{1_H\}$, then $\{1_H\} \subseteq X, Y \subseteq \{1_H\}$.

- $\mathcal{P}_{\mathrm{fin},1}(H)$ is cancellative if and only if H is trivial.

  If $H = \{1_H\}$, also $\mathcal{P}_{\mathrm{fin},1}(H)$ is trivial. If H is non-trivial:
  Case I: $x^2 \notin \{1_H, x\}$ for every non-identity $x \in H$:

  $$\{1, x, x^2, x^3\} = \{1, x^2\}\{1, x\} = \{1, x, x^2\}\{1, x\} \text{ but } \{1, x^2\} \neq \{1, x, x^2\}.$$

  Case II: There exists a non-identity $x \in H$ such that $x^2 \in \{1_H, x\}$:

  $$\{1, x\}\{1, x\} = \{1\}\{1, x\} \text{ but } \{1, x\} \neq \{1\}.$$

[4] A monoid M is **reduced** if $M^\times = \{1_M\}$

Irreducibles, atoms and quarks

Let M be a Dedekind-finite monoid. We say that a non-unit $a \in M$ is:

- an IRREDUCIBLE (of M) if $a \neq xy$ for all non-units $x, y \in M$ that divide a *properly*;
- an ATOM if $a \neq xy$ for all non-units $x, y \in M$;
- a QUARK if it is not properly divided by any non-unit.

ATOM $\Rightarrow$ IRREDUCIBLE and QUARK $\Rightarrow$ IRREDUCIBLE

Irreducibles, atoms and quarks in reduced PMs

Proposition 1 [C. & Tringali, 2025]

Let H be a monoid and let $X, Y, Z \in \mathcal{P}_{\mathrm{fin},1}(H)$. The following hold:

i. X is irreducible in $\mathcal{P}_{\mathrm{fin},1}(H)$ if and only if $\{1_H\} \neq X \neq YZ$ for all $Y, Z \subsetneq X$.

ii. X is irreducible in $\mathcal{P}_{\mathrm{fin},1}(H)$ if and only if it is a quark.

iii. A set $X \in \mathcal{P}_{\mathrm{fin},1}(H)$ is irreducible but not an atom if and only if $X = \{1_H, x\}$ for some $x \in H \setminus \{1_H\}$ such that $x^2 = 1_H$ or $x^2 = x$.

NOTE: Every two element set $\{1_H, x\} \in \mathcal{P}_{\mathrm{fin},1}(H)$ is an irreducible.

Factorizations and minimal factorizations

Given a monoid M, let $\mathscr{I}(M)$ be the set of irreducibles of M.

A FACTORIZATION (INTO IRREDUCIBLES) of an element $x \in M$ is an $\mathscr{I}(M)$-word

$$\mathfrak{a} = a_1 * \cdots * a_n \text{ such that } x = a_1 \cdots a_n.$$

In this case, n is called the LENGTH of the factorization.

For two M-words $\mathfrak{a}, \mathfrak{b}$, define $\mathfrak{a} \sqsubseteq_M \mathfrak{b}$ if, up to associatedness of the letters, $\mathfrak{a}$ is a subword of a permutation of $\mathfrak{b}$. Two M-words $\mathfrak{a}$ and $\mathfrak{b}$ are EQUIVALENT if $\mathfrak{a} \sqsubseteq_M \mathfrak{b} \sqsubseteq_M \mathfrak{a}$.

Accordingly, a factorization $\mathfrak{a}$ of x is a MINIMAL FACTORIZATION if there is no factorization $\mathfrak{b}$ of x with $\mathfrak{b} \sqsubseteq_M \mathfrak{a} \not\sqsubseteq_M \mathfrak{b}$. Two (minimal) factorization of x are EQUIVALENT if they are equivalent as M-words.

Factorizations and minimal factorizations in reduced PMs

Given a monoid H, two $\mathcal{P}_{\mathrm{fin},1}(H)$-words are EQUIVALENT if and only if they are a permutation of each other.

Thus, if $X \in \mathcal{P}_{\mathrm{fin},1}(H)$, we define

- a FACTORIZATION of X as a word

  $$A_1 * \cdots * A_n,$$

  with $A_1, \ldots, A_n$ IRREDUCIBLES, such that

  $$X = A_1 \cdots A_n;$$

- a MINIMAL FACTORIZATION of X as a factorization

  $$A_1 * \cdots * A_n$$

  of X, such that

  $$X \neq A_{\sigma(1)} \cdots A_{\sigma(k)}$$

  for all $k \in [\![1, n-1]\!]$ and every permutation σ of $[\![1, k]\!]$.

We say that a Dedekind-finite monoid M is:

- **factorable** if every non-unit of M admits a factorization into irreducibles;
- **BF** if it is factorable and the set of all factorization lengths of any fixed element is bounded;
- **HF** if it is factorable and the factorizations of any element have all the same length;
- **FF** if it is factorable and each element has finitely many inequivalent factorizations;
- **UF** if it is factorable and any two factorizations of an element are equivalent.

We say that a Dedekind-finite monoid M is:

- **factorable** if every non-unit of M admits a minimal factorization into irreducibles;
- **BmF** if it is factorable and the set of all minimal factorization lengths of any fixed element is bounded;
- **HmF** if it is factorable and the minimal factorizations of any element have all the same length;
- **FmF** if it is factorable and each element has finitely many inequivalent minimal factorizations;
- **UmF** if it is factorable and any two minimal factorizations of an element are equivalent.

Note that, if M is a factorable monoid, then every non-unit $x \in M$ has at least one minimal factorization.

Replacing irreducibles with atoms as the “building blocks” of the decompositions of interest, will result in the (classical) notions of **atomic factorization**, **atomic** monoid, **BF-atomic** monoid, and so on.

Some arithmetical results for reduced PMs

Minimal factorizations were introduced in [Antoniou & Tringali, 2021] where, however, **atoms** are the building blocks of interest. They proved that, for any monoid H

$$\mathcal{P}_{\mathrm{fin},1}(H) \textit{ is atomic if and only if } 1_H \neq x^2 \neq x \textit{ for all } x \in H \setminus \{1_H\}.$$

The next Proposition complements this result:

Proposition 2 [C. & Tringali, 2025]

Given a monoid H:

H is aperiodic[5] $\Leftrightarrow \mathcal{P}_{\mathrm{fin},1}(H)$ is FF(-atomic) $\Leftrightarrow \mathcal{P}_{\mathrm{fin},1}(H)$ is BF(-atomic).

The conditions on H highlight that atomic factorizations and, more generally, “unrestricted factorizations” are not the best choice possible when it comes to “highly non-cancellative” monoids. Minimal factorizations allow us to overcome these limitations.

Proposition 3 [C. & Tringali, 2025]

$\mathcal{P}_{\mathrm{fin},1}(H)$ is FmF for **every** monoid H.

[5] A monoid is **aperiodic** if every non-identity element generates an infinite submonoid.

UmF-ness in reduced PMs, some definitions

Definition [(Almost-)breakable monoid]

A (multiplicative) semigroup (resp., monoid) S is:

- ALMOST-BREAKABLE if, for every $x, y \in S$, $xy \in \{x, y\}$ or $yx \in \{x, y\}$;
- BREAKABLE if $xy \in \{x, y\}$ for all $x, y \in S$.

REMARKS:

- Every breakable semigroup is almost-breakable. The converse need not be true. For instance, let S be the 3-element magma described by the following table:

	x	y	z
x	x	x	x
y	x	y	x
z	z	z	z

  S is an almost-breakable semigroup that is not breakable, since $yz \notin \{y, z\}$.

- Any almost-breakable monoid H is **idempotent**, i.e., $x^2 = x$ for every $x \in H$. Moreover, H is **Dedekind-finite** and **reduced**: if $xy = 1_H$ for some $x, y \in H$, then $y = xy^2 = xy = 1_H$ and hence $x = y = 1_H$.

UmF-ness in reduced PMs, some definitions

Definition [Trivial ideal extension]

Let H and K be disjoint semigroups. We define a “joint extension” of the operations of H and K to a binary operation on $H \cup K$ by taking $xy = yx := y$ for all $x \in H$ and $y \in K$. We denote the magma obtained in this way by $H \oslash K$ and call it the TRIVIAL IDEAL EXTENSION OF K BY H.

REMARK: $H \oslash K$ is a **semigroup** and it is a **monoid** if and only if H is. In this case the identity of $H \oslash K$ is the same as the identity of H.

Theorem 1 [C. & Tringali, 2025]

The following are equivalent for a monoid H:

a. $\mathcal{P}_{\mathrm{fin},1}(H)$ is UmF.
b. $H \setminus H^\times$ is an almost-breakable subsemigroup of H, $H^\times$ has order ≤ 2, $H = H^\times \oslash (H \setminus H^\times)$, and $\mathcal{P}_{\mathrm{fin},1}((H \setminus H^\times) \cup \{1_H\})$ is UmF.

The (unit-)cancellative case

Corollary 1 [C. & Tringali, 2025]

Let H be either a group or a monoid whose non-units do not form an almost-breakable semigroup. Then,

$$\mathcal{P}_{\mathrm{fin},1}(H) \text{ is UmF} \Leftrightarrow H = \{1_H\} \text{ or } H \simeq (\mathbb{Z}_2, +).$$

Remarks:

- If a monoid H is **cancellative**, **unit-cancellative**[6], or **acyclic**[7], there is no idempotent element of H different from the identity and $H \setminus H^\times$ cannot be an almost-breakable semigroup whenever it is non-empty.

- Corollary 1 extends a previous result in [Antoniou & Tringali, 2021]:

  $\mathcal{P}_{\mathrm{fin},1}(H)$ is UmF-atomic if and only if H is trivial.

  Recall, in fact, that $\mathcal{P}_{\mathrm{fin},1}(H)$ is atomic if and only if $1_H \neq x^2 \neq x$ for every non-identity element of H.

[6] A monoid H is **unit-cancellative** if $xu \neq x \neq ux$ for all $u, x \in H$ such that $u$is a non-unit.

[7] A monoid H is **acyclic** if $x \neq uxv$ for all $u, v, x \in H$ such that u or v is a non-unit.

Focus on the almost breakable case

In light of Theorem 1, characterizing the UmF-ness of $\mathcal{P}_{\text{fin},1}(H)$ comes down to the case when H is almost-breakable.

Proposition 4 [C. & Tringali, 2025]

The ground monoid H being almost-breakable is NOT a sufficient condition for $\mathcal{P}_{\text{fin},1}(H)$ to be UmF.

EXAMPLE:

Let H be the unitization of the 4-element magma defined by this table:

	x	y	z	w
x	x	x	x	x
y	x	y	x	y
z	z	z	z	z
w	z	w	z	w

.

H is an almost-breakable monoid and, since $yz = x$ and $wx = z$, we can construct two inequivalent minimal factorizations of $\{1_H, x, y, z, w\}$:

$$\{1_H, x, y, z, w\} = \{1_H, y\}\{1_H, z\}\{1_H, w\} = \{1_H, w\}\{1_H, x\}\{1_H, y\}.$$

The commutative case

Proposition 5 [C. & Tringali, 2025]

If H is a breakable monoid, then $\mathcal{P}_{\mathrm{fin},1}(H)$ is UmF.

Since

a commutative monoid is almost-breakable if and only if it is breakable,

we obtain a complete characterization of UmF-ness of $\mathcal{P}_{\mathrm{fin},1}(H)$, when H is commutative.

Theorem 2 [C. & Tringali, 2025]

The following are equivalent for a **commutative** monoid H:

a. $\mathcal{P}_{\mathrm{fin},1}(H)$ is UmF.

b. $H \setminus H^\times$ is a breakable subsemigroup of H, $H^\times$ has order ≤ 2, and $H = H^\times \oslash (H \setminus H^\times)$.

Prospects for future research

Proposition 6 [C. & Tringali, 2025]

If $\mathcal{P}_{\text{fin},1}(H)$ is HmF, then $x^3 \in \{1_H, x, x^2\}$ for every $x \in H$.

Some questions:

- Can we characterize HmF-ness for $\mathcal{P}_{\text{fin},1}(H)$?
- Can we improve the characterization of UmF-ness?
- Can we compute arithmetical invariants (sets of minimal leghts, minimal elasticity, ...) on $\mathcal{P}_{\text{fin},1}(H)$, for special choices of H?

THANK YOU FOR YOUR ATTENTION!

L. C., S. TRINGALI, On the arithmetic of power monoids, *J. Algebra*, in press.