Isomorphism problems for ideals of numerical semigroups

Recent Trends in the Theory of Power Semigroups

Pedro A. García Sánchez [a]

School of Mathematical Sciences, Hebei Normal University, 21 September 2025

Departamento de Álgebra and IMAG, Universidad de Granada

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Numerical semigroups

Let $\mathbb{N}$ denote the set of non-negative integers

A *numerical semigroup* is a submonoid of $(\mathbb{N}, +)$ with finite complement in $\mathbb{N}$

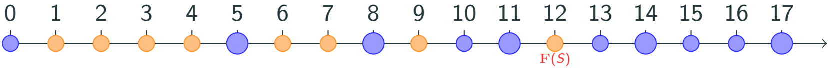
Notable elements

$S = \{0, 5, 8, 10, 11, 13, \rightarrow\}$ (blue dots); $S^* = S \setminus \{0\}$

Atoms, minimal generators: $S^* \setminus (S^* + S^*)$ (big blue dots)

Gaps: $\mathrm{G}(S) = \mathbb{N} \setminus S$ (orange dots)

Frobenius number: $\mathrm{F}(S) = \max(\mathbb{Z} \setminus S)$



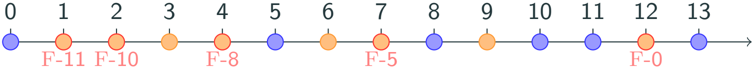
Genus: $\mathrm{g}(S) = |\,\mathrm{G}(S)|$

Sporadic (left) elements: $S \cap [0, \mathrm{F}(S)]$, its cardinality is denoted by $\mathrm{n}(S)$

We will use the notation $\langle A \rangle$ for the monoid spanned by A

A lower bound for the genus; symmetry

Let S be a numerical semigroup



As $\mathrm{F}(S) - x \notin S$, for every $x \in S$, we deduce that $\mathrm{n}(S) \leq \mathrm{g}(S)$.

Thus, $\mathrm{F}(S) + 1 = \mathrm{n}(S) + \mathrm{g}(S) \leq 2\,\mathrm{g}(S)$

$$\mathrm{g}(S) \geq \frac{\mathrm{F}(S) + 1}{2}$$

The semigroup S is *symmetric* if $\mathrm{g}(S) = \frac{\mathrm{F}(S)+1}{2}$; equivalently,

$x \in \mathbb{Z} \setminus S$ implies $\mathrm{F}(S) - x \in S$

Pseudo-symmetric numerical semigroups

If S is a symmetric numerical semigroup, then $\mathrm{F}(S)$ is odd

We say that S is a *pseudo-symmetric* numerical semigroup if $\mathrm{F}(S)$ is even and for every $x \in \mathbb{Z} \setminus S$, $x \neq \frac{\mathrm{F}(S)}{2}$, we have $\mathrm{F}(S) - x \in S$

S is pseudo-symmetric if and only if $\mathrm{g}(S) = \frac{\mathrm{F}(S)+2}{2}$

Thus, symmetric and pseudo-symmetric numerical semigroups are those numerical semigroups that have less possible genus if the Frobenius number is fixed

Equivalently, these semigroups are maximal (with respect to set inclusion) in the set of all numerical semigroups having a fixed Frobenius number

Irreducible numerical semigroups

A numerical semigroup S is *irreducible* if it cannot be expressed as the intersection of two numerical semigroups properly containing it

It is not difficult to show that S is irreducible if and only if it is maximal in the set of numerical semigroups with Frobenius number $\mathrm{F}(S)$

Equivalently, in the set of numerical semigroups not containing $\mathrm{F}(S)$

Thus, S is irreducible if and only if it is either symmetric or pseudo-symmetric

Pseudo-Frobenius numbers and irreducibles

Let S be a numerical semigroup

Given a and b integers, we write $a \leq_S b$ if $b - a \in S$

The set of *pseudo-Frobenius* numbers of S is

$$\mathrm{PF}(S) = \mathrm{Maximals}_{\leq_S}(\mathbb{Z} \setminus S) = \{x \in \mathbb{Z} \setminus S : x + S^* \subseteq S\}$$

S is symmetric if and only if $\mathrm{PF}(S) = \{\mathrm{F}(S)\}$

S is pseudo-symmetric if and only if $\mathrm{PF}(S) = \{\frac{\mathrm{F}(S)}{2}, \mathrm{F}(S)\}$

The cardinality of $\mathrm{PF}(S)$ is the *type* of S, denoted $\mathrm{t}(S)$

Special gaps and irreducible numerical semigroups

A gap g of S is a *special gap* if $S \cup \{g\}$ is a numerical semigroup

$$\mathrm{SG}(S) = \{g \in \mathrm{PF}(S) : 2g \in S\}$$

S is irreducible if and only if $|\mathrm{SG}(S)| \leq 1$

$|\mathrm{SG}(S)|$ is the number of "unitary extensions" of S

Ideals of a numerical semigroup

For $A, B \subseteq \mathbb{Z}$, we write $A + B = \{a + b : a \in A, b \in B\}$

A (relative) **ideal** of a numerical semigroup S is a non-empty set I of integers such that

- $I + S \subseteq I$
- $z + I \subseteq S$ for some integer z (equivalently, $\min(I)$ exists)

There exists $X \subseteq \mathbb{Z}$ such that $I = X + S$, we can take X to be a finite set of integers

The set X is called a **set of generators** of I, and it is a **minimal set of generators** of I if no proper subset of X generates I

The minimal set of generators of I is unique and equals $\text{Minimals}_{\leq_S}(I)$, where

$$a \leq_S b \text{ if } b - a \in S$$

If I, J are ideals of S, then $I + J$, $I \cap J$, $I \cup J$ are ideals of S

The monoid of ideals of a numerical semigroup

Let S be a numerical semigroup and $\mathcal{I}(S)$ the set of ideals of **proper ideals** of S (ideals included in S)

The set $\mathcal{I}(S)$ is a monoid with respect to the operation $+$, the sum of ideals

If $\mathcal{I}(S)$ and $\mathcal{I}(T)$ are isomorphic as monoids, then $S = T$

This is because

- the set of principal ideals of S, $\mathfrak{P}(S)$ is a divisor closed submonoid of $\mathcal{I}(S)$ that is isomorphic to S
- every (non-trivial) divisor-closed submonoid of $\mathfrak{I}(S)$ contains $\mathfrak{P}(S)$
- isomorphisms between monoids send divisor-closed submonoids to divisor-closed submonoids

The power monoid isomorphism problem and ideals

Let S and T be numerical semigroups

Every isomorphism between the power monoids $(\mathcal{P}(S), +)$ and $(\mathcal{P}(T), +)$ restricts to an isomorphism between the monoids $(\mathcal{I}(S), +)$ and $(\mathcal{I}(T), +)$

Thus, if S and T are globally isomorphic, then $S = T$

These results can be generalized to a more general setting of monoids (strongly Archimedean, cancellative, and reduced)

The ideal class monoid

Let S be a numerical semigroup

Let $\mathcal{I}_r(S)$ be the set of (relative) ideals of S

We write $I \sim J$ if there exists $z \in \mathbb{Z}$ such that $I = z + J$

The **ideal class monoid** of S is

$$\mathcal{C}\ell(S) = \mathcal{I}_r(S)/{\sim}$$

Addition is defined as $[I] + [J] = [I + J]$

The set of normalized ideals

Let $\mathcal{I}_0(S) = \{I \in \mathcal{I}_r(S) : \min(I) = 0\}$, the set of normalized ideals of S

It follows easily that

$$\mathcal{C}\ell(S) \cong \mathcal{I}_0(S);\ [I] \mapsto -\min(I) + I$$

For $I \in \mathcal{I}_0(S)$, there exists $g_1, \ldots, g_k \in \mathrm{G}(S) = \mathbb{N} \setminus S$ such that

$$I = \{0, g_1, \ldots, g_k\} + S$$

Moreover, $\{g_1, \ldots, g_k\}$ can be taken to be an anti-chain with respect to $\leq_S$

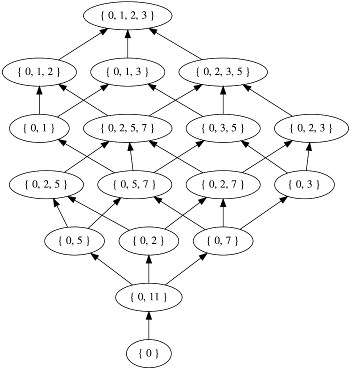
Thus, $0, g_1, \ldots, g_k$ are the minimal generators of I

Thus, the cardinality of $\mathcal{C}\ell(S)$ equals the number of antichains in $(\mathrm{G}(S), \leq_S)$

Hasse diagram of normalized ideals wrt inclusion

Let S be a numerical semigroup with multiplicity m, that is, $m = \min(S \setminus \{0\})$

- $\min_{\subseteq}(\mathcal{I}_0(S)) = S$
- $\max_{\subseteq}(\mathcal{I}_0(S)) = \mathbb{N}$
- Maximal non-trivial ideals: $\{0, 1, \ldots, i-1, i+m, i+1, \ldots, m-1\} + S$
  $|\operatorname{Maximals}_{\subseteq}(\mathcal{I}_0(S) \setminus \{\mathbb{N}\})| = m - 1$
- Minimal non-trivial ideals: $\{0, f\} + S$ with $f \in \operatorname{PF}(S)$
  $|\operatorname{Minimals}_{\subseteq}(\mathcal{I}_0(S) \setminus \{S\})| = \operatorname{t}(S)$
- Height equals $|\operatorname{G}(S)| + 1$, the genus of S plus one



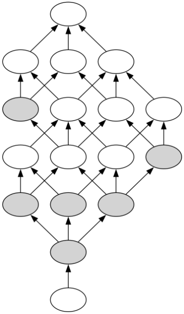
Hasse diagram of $(\mathcal{I}_0(\langle 4, 6, 9\rangle), \subseteq)$, nodes labelled by sets of minimal generators

Can we fully recover S from $(\mathcal{I}_0(S), \subseteq)$?

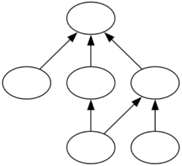
Idea: $\{0, g\} + S \subseteq \{0, g'\} + S$ if and only if $g' \leq_S g$

Thus, if we can tell apart the ideals of the form $\{0, g\} + S$, with $g \in \mathsf{G}(S)$, then we recover $(\mathsf{G}(S), \leq_S)$

Result: The number of non-zero minimal generators of I is precisely the number of elements in the poset $(\mathcal{I}_0(S), \subseteq)$ covered by I



$(\mathcal{I}_0(S), \subseteq)$



$(\mathsf{G}(S), \leq_S)$

How to fully reconstruct S from $(\mathrm{G}(S), \leq_S)$?

Idea: counting "divisors"

For $h \in \mathrm{G}(S)$, set $\mathrm{nd}(h) = |\{h' \in \mathrm{G}(S) : h' \leq_S h\}|$

Result: For every $h, h' \in \mathrm{G}(S)$, $h \leq h'$,

$$|S \cap [h, h']| = \mathrm{nd}(h') - \mathrm{nd}(h)$$

In a run of gaps, the number of divisors remains constant

After a run of elements of S, the number of divisors increases by the length of the run

Conclusion: If S and T are numerical semigroups such that the poset $(\mathcal{I}_0(S), \subseteq)$ is isomorphic to $(\mathcal{I}_0(T), \subseteq)$, then $S = T$

Monoid isomorphism problem

If S and T are numerical semigroups such that the monoid $(\mathcal{I}_0(S), +)$ is isomorphic to $(\mathcal{I}_0(T), +)$, can we ensure that $S = T$?

For $I, J \in \mathcal{I}_0(S)$, write $I \preceq J$ if there exists $K \in \mathcal{I}_0(S)$ such that $I + K = J$

Some observations

1. Oversemigroups of S are precisely the ideals I of $\mathcal{I}_0(S)$ such that $I + I = I$ (idempotent)
2. Unitary extensions of S are minimal idempotents with respect to $\preceq$ (idempotent quarks)
3. The genus of S (the cardinality of $\mathrm{G}(S)$) is the maximum k such that there exists a chain $I_0 \prec I_1 \prec \cdots \prec I_k$ in $\mathcal{I}_0(S)$
4. For $E \in \mathcal{I}_0(S)$, with $E + E = E$ (an oversemigroup), the set

$$C_E = \{I \in \mathcal{I}_0(S) : I + E = I\} = \mathcal{I}_0(E)$$

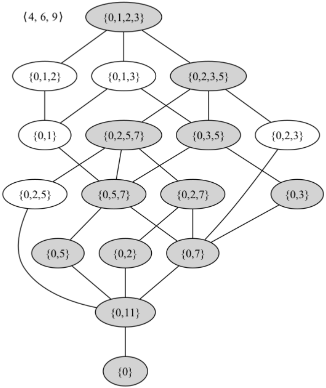
Monoid isomorphism problem (solution)

If S and T are numerical semigroups such that the monoid $(\mathcal{I}_0(S), +)$ is isomorphic to $(\mathcal{I}_0(T), +)$, then $S = T$?

1. Apply induction on the genus: S and T have the same genus and the same unitary extensions
2. The intersection of all the unitary extensions of $S \neq \mathbb{N}$ equals S or $S \cup \{\mathrm{F}(S)\}$ (if S is irreducible)
3. S is irreducible if and only if $\mathcal{I}_0(S)$ has at most two quarks; if $S \neq \mathbb{N}$,
    - one quark means symmetric
    - two quarks translates pseudo-symmetry
4. In the irreducible case, we recover $\mathrm{F}(S)$ from the genus of S

The other (unsolved) poset isomorphism

If S and T are numerical semigroups such that the poset $(\mathcal{I}_0(S), \preceq)$ is isomorphic to $(\mathcal{I}_0(T), \preceq)$, can we ensure that $S = T$?



Hasse diagram of $\mathcal{I}_0(\langle 4, 6, 9\rangle)$; nodes labelled with sets of minimal generators gray nodes are idempotents (oversemigroups)

The multiplicity three case

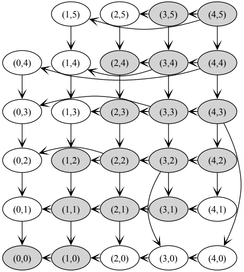
The multiplicity of S is the smallest positive integer in S, denoted by $\mathrm{m}(S)$

For $I \in \mathcal{I}_0(S)$, the Apéry set of I is

$$\mathrm{Ap}(I) = \{i \in I : i - \mathrm{m}(S) \notin I\} = \{0, i_1, \ldots, i_{\mathrm{m}(S)-1}\}$$

Every $i_j = k_j\,\mathrm{m}(S) + j$ for some $k_j \in \mathbb{N}$: $(k_1, \ldots, k_{\mathrm{m}(S)-1})$ are the Kunz coordinates of I

If $\mathrm{m}(S) = 3$, then the Kunz coordinates of I are (k_1, k_2) (fulfilling a known set of inequalities depending on the Kunz coordinates of S)



Hasse diagram of $(\mathcal{I}_0(\langle 3, 14, 17\rangle), \preceq)$; nodes labelled with Kunz coordinates; gray nodes are idempotents (oversemigroups)

The multiplicity three case (cont.)

The depth of I is the largest k such that there exists a chain $I = I_0 \prec I_1 \prec \cdots \prec I_k = \mathbb{N}$ in $\mathcal{I}_0(S)$

For multiplicity 3, the depth of I is the sum of its Kunz coordinates

Also, $(\mathcal{I}_0(S), \preceq)$ is a lattice and has at most three quarks

With all this information we can recover the Kunz coordinates of S and thus S itself; in particular, for multiplicity three the isomorphism problem for the poset $(\mathcal{I}_0(S), \preceq)$ is solved

When is the poset $(\mathcal{I}_0(S), \preceq)$ a lattice?

We know when $\preceq$ coincides with $\subseteq$ in $\mathcal{I}_0(S)$ (and thus in these cases the poset is a lattice):

$$S \in \{\langle 3,4\rangle, \langle 3,4,5\rangle, \langle 3,5\rangle, \langle 3,5,7\rangle\} \cup \{\langle 2, 2k+1\rangle : k \in \mathbb{N}\}.$$

We also know that if the multiplicity of S is three, then $(\mathcal{I}_0(S), \preceq)$ a lattice

By studying how the posets $(\mathcal{I}_0(S), \preceq)$ and $(\mathcal{I}_0(S \setminus \{a\}), \preceq)$ with a a minimal generator larger than the Frobenius number of S are related, we can show that for multiplicity four the poset $(\mathcal{I}_0(S), \preceq)$ is a lattice

If S has multiplicity larger than five, then the poset $(\mathcal{I}_0(S), \preceq)$ is not a lattice

Conclusion: The poset $(\mathcal{I}_0(S), \preceq)$ is a lattice if and only if S has multiplicity at most four

Related problems

Irreducible elements

Detect irreducibles with respect to the operations in $\mathcal{I}_0(S)$: $+$, $\cap$, $\cup$

Decompositions into irreducibles (factorizations)

Quotient of ideals

We can compute $I + J$, $I \cap J$ and $I \cup J$ for $I, J \in \mathcal{I}_0(S)$, from the Kunz coordinates of I and J

A better understanding of the Kunz coordinates of

$$I - J = \{z \in \mathbb{Z} : z + J \subseteq I\}$$

would be useful to study reflexive ideals

Associated numerical sets of a numerical semigroup

Let S be a numerical semigroup and let T be a cofinite subset of $\mathbb{N}$ with $0 \in T$

We say that T is associated to S if

$$S = \{x \in \mathbb{N} : x + T \subseteq T\}$$

It easily follows that T is associated to S if and only if

- $T \in \mathcal{I}_0(S)$, and
- $T - T = S$

There is a one-to-one correspondence between the set of associated numerical sets of S and the set of partitions associated to S (partitions whose hook sets are $\mathrm{G}(S)$)

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Thank you for your attention!

Find out more at numerical-semigroups.github.io