Maximal Common Divisors in Power Monoids

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Recent Trends in the Theory of Power Semigroups

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Outline

Preliminary

For a commutative ring R with identity, we say that

1. R is an integral domain if for all $r, s \in R$ with $rs = 0$ either $r = 0$ or $s = 0$.
2. R is atomic if every nonzero element factors into finitely many irreducibles (also called atoms).
3. R is an IDF domain if every nonzero element has only finitely many irreducible divisors (up to associate).

Motivation and Relevance

Remark. Anderson, Anderson, and Zafrullah, 1990
Factorization in integral domains (FIT)

FIT is a highly influential paper in the arena of factorization theory.

1. It introduces the following generalizations of a UFD:
   - a finite factorization domain (FFD) is an atomic domain whose nonzero elements have only finitely many factorizations and
   - a bounded factorization domain (BFD) is an atomic domain whose nonzero elements have only finitely many factorization lengths.
2. It connects BFDs/FFDs with Noetherian/Krull domains:
   - Noetherian domains are proved to be BFDs
   - Krull domains are proved to be FFDs.
3. It shows that the class of BFDs is contained in that of ACCP domains.
4. It shows that the class of FFDs is precisely that consisting of all atomic IDF domains.

Motivation and Relevance (continuation)

Remark. In FIT, the authors left exactly two open questions.

Question (FIT, Question 1)

Given an integral domain R that is atomic, can we guarantee that $R[x]$ is also atomic?

Question (FIT, Question 2)

Given an integral domain R that is an IDF domain, can we guarantee that $R[x]$ is also an IDF domain?

Remark. Although the answers of both questions are negative, we can use maximal common divisors (MCDs) to provide natural sufficient conditions where the answers are positive.

Our Goal Today is Twofold

1. Discussing the answers to Questions 1 and 2 in more detail.
2. Providing the parallel of these questions in the setting of power monoids.

Power Semigroups

Definition. Let S be a semigroup, which is multiplicatively written. Under the product

- $X \cdot Y := \{xy : (x, y) \in X \times Y\}$ for any $X, Y \in \mathcal{P}(S)$,
- $\mathcal{P}(S) := 2^S \setminus \{\emptyset\}$ is called the power semigroup of S, and
- $\mathcal{P}_{\text{fin}}(S) := \{X \in \mathcal{P}(S) : |X| < \infty\}$ is called the finitary power semigroup of S.

Moreover, if S is a monoid with identity element 1, then

- $\mathcal{P}(S)$ is also a monoid with identity element $\{1\}$ and
- $\mathcal{P}_{\text{fin}}(S)$ is a submonoid of $\mathcal{P}(S)$.

Recent Work on Power Monoids

Remarks. In 2018, Fan and Tringali first studied the arithmetic of factorizations inside the prototypical power monoid $\mathcal{P}_{\text{fin}}(\mathbb{N}_0)$.

Their work has motivated a series of subsequence papers in power semigroups/monoids, including the following.

- Antoniou and Tringali (2021)
  On the arithmetic of power monoids and sumsets in cyclic groups
- Tringali and Yan (2023)
  A conjecture by Bienvenu and Geroldinger on power monoids
- Tringali and Yan (2025)
  On power monoids and their automorphisms
- García-Sánchez and Tringali (2025)
  Semigroups of ideals and isomorphism problems
- Cossu and Tringali (2025)
  On the arithmetic of power monoids

Commutative Setting

Remark. Further recent papers fully dedicated to commutative power monoids have appeared recently, including the following.

- Bienvenu and Geroldinger (2022)
  On algebraic properties of power monoids of numerical monoids
- Gonzalez, Li, Rabinovitz, Rodriguez, Tirador (2023)
  On the atomicity of power monoids of Puiseux monoids
- Aggarwal, Gotti, Lu (2024)
  On primality and atomicity of numerical power monoids
- Dani, Gotti, Hong, Li, Schlessinger (2024)
  On finitary power monoids of linearly orderable monoids
- Blitz, Gotti, Han, Liang (2025)
  Maximal common divisors and power monoids

A 30-Minute Convention. From now on, all monoids in this presentation are tacitly assumed to be commutative.

Ascent of Algebraic Properties – Polynomial Extensions

Definition. We say that an algebraic property $\mathcal{P}$ ascends to polynomial extensions if for any ring R satisfying the property $\mathcal{P}$ the polynomial ring $R[x]$ also satisfies the property $\mathcal{P}$.

Remark. The ascent of algebraic properties to polynomial extensions is central in the study of (commutative) ring theory. Here are two fundamental examples.

Theorem (Gauss's Theorem)

If R is a UFD, then $R[x]$ is a UFD.

Theorem (Hilbert Basis Theorem)

If R is a Noetherian domain, then $R[x]$ is a Noetherian domain.

Polynomial Rings – Ascent of Atomicity

Definitions [Cohn, 1968]

- A monoid is called atomic if every non-invertible element factors into finitely many atoms.
- An integral domain is called atomic if its multiplicative monoid is atomic.

Examples

- Every monoid/domain satisfying the ACCP is atomic.
- Every Krull and Noetherian monoid/domain is atomic.

Question (Anderson-Anderson-Zafrullah, 1990)

Does the property of being atomic ascend to polynomial rings?

The question was first answered by Roitman (negatively).

Theorem (Roitman, 1993)

There exists an integrally closed atomic domain R such that $R[x]$ is not atomic.

Polynomial Rings – Ascent of the IDF Property

Definitions [Grams and Warner 1975]

- A monoid is called an IDF monoid if every element is divisible by only finitely many atoms (up to associate).
- An integral domain is called an IDF domain if its multiplicative monoid is an IDF monoid.

Examples

- Every monoid/domain satisfying the finite factorization property.
- Every monoid/domain having no atoms (they are called antimatter).

Question (Anderson-Anderson-Zafrullah, 1990)

Does the IDF property ascend to polynomial rings?

The question was first answered by Malcolmson and Okoh (negatively).

Theorem (Malcolmson-Okoh, 2009)

Every countable integral domain embeds into a countable antimatter domain S such that $S[x]$ is not IDF domain.

Maximal Common Divisors

Definition. Let M be a monoid, and let S be a nonempty subset of M.

- A common divisor $d \in M$ of S is called a greatest common divisor (GCD) if d is divisible by any common divisor of S.
- A common divisor $d \in M$ of S is called a maximal common divisor (MCD) if the only common divisors of S/d are the units of M.

Remark. Although every GCD is an MCD, the converse does not hold.

Example

1. Consider the numerical monoid $N := \{0\} \cup \mathbb{N}_{\geq 2}$.
2. The common divisors of $S := \{5, 6\}$ are $0, 2$, and 3.
3. It is clear that 0 is not an MCD of S (and so 0 is not a GCD).
4. As $2 \nmid_N 3$ and $3 \nmid_N 2$, neither 2 nor 3 are GCDs of S.
5. Note that $S - 2 = \{3, 4\}$ and that 0 is the only common divisor of $\{3, 4\}$. So 2 is an MCD of S.
6. Similarly, the only common divisor of $S - 3 = \{2, 3\}$ in N is 0, and so 3 is an MCD of S in N.

Back to the Ascent of Atomicity – A Positive Answer

Definition. Let M be a monoid, and let S be a nonempty finite subset of M.

- M is called an MCD monoid if every nonempty finite subset of M has an MCD.
- An integral domain R is called an MCD domain if its multiplicative monoid R^* is an MCD monoid.

Examples

- Every GCD monoid/domain is an MCD monoid/domain.
- Every monoid/domain satisfying the ACCP is an MCD monoid/domain.

Theorem (Roitman 1993)

Let R be an MCD domain. If R is atomic, then $R[x]$ is atomic.

Back to the Ascent of the IDF property – A Positive Answer

Definition. Let M be a monoid, and let S be a nonempty subset of M.

- M is called an MCD-finite monoid if every nonempty finite subset of M has only finitely many MCDs (up to associates).
- An integral domain R is called an MCD-finite domain if its multiplicative monoid R^* is an MCD monoid.

Examples

- Pre-Schreier monoid/domain is MCD-finite.
- Every monoid/domain having the FF property is MCD-finite.

Theorem (Eftekhari-Khorsandi, 2018)

Let R be an MCD-finite domain. If R is an IDF domain, then $R[x]$ is an IDF domain.

Ascent of Algebraic Properties – Power Monoids

Definition. We say that an algebraic property $\mathcal{P}$ ascends to power monoids if for any monoid M satisfying the property $\mathcal{P}$ the finitary power monoid of M also satisfies the property $\mathcal{P}$.

Question

1. *Does the property of being atomic ascend to power monoids?*
2. *Does the IDF property ascend to power monoids?*
3. *If not, can we use MCDs to provide sufficient conditions for positive answers?*

Ascent of Atomicity – Power Monoid

As for polynomials, atomicity does not ascend to power monoids.

Theorem (Gonzalez-Li-Rabinovitz-Rodriguez-Tirador, 2023)

There exists an atomic monoid M with a non-atomic power monoid.

As for polynomials, atomicity ascends to power monoids if MCDs exist.

Theorem (Jiya-Gotti-Hong-Li-Schlessinger, 2024)

For a monoid M, the following conditions are equivalent.

- *The monoid M is atomic and MCD.*
- *$\mathcal{P}_{fin}(M)$ is atomic and MCD monoid.*
- *$\mathcal{P}_{fin}(M)$ is atomic.*

Let R be an MCD domain. If R is atomic, then $R[x]$ is atomic.

Proposition

The MCD property ascends to power monoids.

Ascent of the IDF Property – Power Monoid

As for polynomials, the IDF property does not ascend to power monoids.

Theorem (Jiya-Gotti-Hong-Li-Schlessinger, 2024)

There exists a monoid M with the IDF property whose finitary power monoid is not IDF.

As for polynomials, the IDF property ascends to power monoids on the class of MCD-finite monoids.

Theorem (Blitz-Gotti-Han-Liang, 2025)

Let M be a cancellative and torsion-free monoid M, the following conditions are equivalent.

- *M is an MCD-finite IDF monoid.*
- *$\mathcal{P}_{fin}(M)$ is an MCD-finite IDF monoid.*
- *$\mathcal{P}_{fin}(M)$ is an IDF monoid.*

Theorem (Blitz-Gotti-Han-Liang, 2025)

The MCD-finite property ascends to power monoids.

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End of Presentation

THANK YOU!