On power monoids and their automorphisms

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Recent Trends in the Theory of Power Semigroups

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Semigroups and monoids

- A (multiplicative) *semigroup* $(S, \cdot)$ is a set endowed with an associative binary operation $(x, y) \mapsto x \cdot y = xy$. Generally, we will use multiplicative notation but depending on the semigroup, we will also use addition.
- We call $x \in S$ *cancellative* if $xa = xb \implies a = b$ and $ax = bx \implies a = b$ for every $a, b \in S$ and we say that S is cancellative if every element is cancellative.
- A *monoid* H is a semigroup with a (unique) identity element $1 = 1_H$ (or $0 = 0_H$) .
- Every commutative, cancellative monoid H is contained in an abelian group, the smallest of which is the *quotient group* of H,

$$\mathrm{q}(H) = \{ab^{-1} : a, b \in H\}.$$

  E.g. $\mathrm{q}((\mathbb{N}_0, +)) = (\mathbb{Z}, +)$.

Power semigroups

- The *large power semigroup* of a semigroup S, denoted by $\mathcal{P}(S)$, is the family of all non-empty subsets of S endowed with set multiplication

$$X \cdot Y = \{x \cdot_S y : x \in X, y \in Y\}.$$

- If H is a monoid, then $\mathcal{P}(H)$ is a monoid with identity element $\{1_H\}$ and we call $\mathcal{P}(H)$ the *large power monoid* of H.

Power semigroups

- Let S be a semigroup and H a monoid. We define the following subsemigroups of $\mathcal{P}(S)$ (resp. of $\mathcal{P}(H)$).
- $\mathcal{P}_{\text{fin}}(S) = \{X \in \mathcal{P}(S) : |X| < \infty\}$, the *finitary power semigroup* of S.
- $\mathcal{P}_{\times}(H) = \{X \in \mathcal{P}(H) : X \cap H^{\times} \neq \emptyset\}$, the *restricted large power monoid* of H.
- $\mathcal{P}_{1}(H) = \{X \in \mathcal{P}(H) : 1_H \in X\}$, the *reduced large power monoid* of H.
- $\mathcal{P}_{\text{fin},\times}(H) = \mathcal{P}_{\text{fin}}(H) \cap \mathcal{P}_{\times}(H)$, the *restricted finitary power monoid* of H.
- $\mathcal{P}_{\text{fin},1}(H) = \mathcal{P}_{\text{fin}}(H) \cap \mathcal{P}_{1}(H)$, the *reduced finitary power monoid* of H.

Automorphisms of power semigroups

Problem

For a given semigroup S, determine the automorphism group $\mathrm{Aut}(\mathcal{P}_\star(S))$.

- Automorphisms help us to understand algebraic objects in a better way.
- E.g. Field automorphisms (Galois Theory), group automorphisms and graph automorphisms.
- Let $f : S \to S$ be an automorphism. We can canonically extend f to an automorphism F of $\mathcal{P}_\star(S)$ by mapping X to

$$F(X) = \{f(x) : x \in X\}.$$

  F is called the *augmentation* of f.
- We always have a canonical embedding $\mathrm{Aut}(S) \hookrightarrow \mathrm{Aut}(\mathcal{P}_\star(S))$. When is this embedding an isomorphism?

Recent work

- The map $\mathrm{rev} : \mathcal{P}_{\mathrm{fin},0}(\mathbb{N}_0) \to \mathcal{P}_{\mathrm{fin},0}(\mathbb{N}_0)$

$$X \mapsto \max(X) - X$$

is an automorphism. For example

$$\mathrm{rev}(\{0, 1, 2, 4\}) = \{0, 2, 3, 4\}.$$

Theorem (Tringali-Yan, 2025)

The only non-trivial automorphism of $\mathcal{P}_{\mathrm{fin},0}(\mathbb{N}_0)$ is the reversion map.

Theorem (Tringali-Wen, 202?)

$\mathrm{Aut}(\mathcal{P}_{\mathrm{fin}}(\mathbb{Z})) \simeq C_2 \times \mathrm{Dih}_\infty$, where Dih_∞ is the infinite dihedral group on two generators.

Tackling automorphisms of $\mathcal{P}_{\text{fin},1}(H)$

- The following lemma is crucial for studying automorphisms of $\mathcal{P}_{\text{fin},1}(H)$.

Lemma (Tringali-Yan, 202?)

Let H, K be monoids and let $f : \mathcal{P}_{\text{fin},1}(H) \to \mathcal{P}_{\text{fin},1}(K)$ be an isomorphism. Then f maps 2-element sets to 2-element sets.

- This yields a canonical bijection $g : H \to K$, satisfying $f(\{1_H, x\}) = \{1_K, g(x)\}$ and $g(1_H) = 1_K$. The map g is called the *pullback* of f.
- The pullback is NOT necessarily a homomorphism.
- $\{1_H, a\} \cdot \{1_H, b\} = \{1_H, a, b, ab\}$ and $\{1_H, ab\}$ do not have an apparent connection.

Tackling automorphisms of $\mathcal{P}_{\mathrm{fin},1}(H)$

- The following lemma is useful, when working with finite monoids.

Lemma

Let H and K be monoids, let $f : \mathcal{P}_{\mathrm{fin},1}(H) \to \mathcal{P}_{\mathrm{fin},1}(K)$ be an isomorphism with pullback g and let M be a finite submonoid of H. Then $f(M)$ is a finite submonoid of K and we have $f(M) = g[M]$.

- Note that if G is a finite abelian group, then the finite submonoids of G are precisely its subgroups.

Tackling automorphisms of $\mathcal{P}_{\text{fin},1}(H)$

Proof

A set $X \in \mathcal{P}_{\text{fin},1}(H)$ is idempotent (i.e. we have $X \cdot X = X$) if and only if X is a (finite) submonoid of H. Since f maps idempotents to idempotents, we obtain that $f(M)$ is a (finite) submonoid of K for every finite submonoid M of H. Moreover, for every non-trivial $x \in H$, we have

$$\{1, x\} \cdot M = M$$

if and only if $x \in M$. This implies that

$$\{1, g(x)\} \cdot f(M) = f(M)$$

if and only if $g(x) \in f(M)$ if and only if $x \in M$ and thus $f(M) = g[M]$. □

Finite abelian groups

- We will always write finite abelian groups additively and for a finite abelian group G, we switch to the notation $\mathcal{P}_0(G)$.
- We denote by C_n the cyclic group of order n, i.e. $C_n \simeq (\mathbb{Z}/n\mathbb{Z}, +)$.

Theorem (Classification of finite abelian groups)

Let G be a non-trivial finite abelian group. Then G can be uniquely (up to isomorphism) written as

$$C_{n_1} \oplus \ldots \oplus C_{n_r},$$

where $r \in \mathbb{N}, n_1, \ldots n_r \in \mathbb{N}_{\geq 2}$ and $n_1 \mid n_2 \mid \ldots \mid n_r$. The integer r is called the *rank* of G.

- It turns out that in our setting, the pullback behaves nicely.

Theorem (Tringali-Yan, 202?)

Let G, H be torsion groups and let $f : \mathcal{P}_{\text{fin},1}(G) \to \mathcal{P}_{\text{fin},1}(H)$ be an isomorphism. Then the pullback of f is an isomorphism.

Starting with cyclic groups

Proposition

Let $G \simeq C_p$, where $p \geq 3$ is a prime number. Then $\mathrm{Aut}(\mathcal{P}_0(G)) \simeq \mathrm{Aut}(G)$.

Proof

Let f be an automorphism of $\mathcal{P}_0(G)$ with pullback g. We want to show that f is the augmentation of g. Hence we can assume that g is the identity with the aim to show that f is the identity. Let $a \in G$ be nonzero. Then

$$(p-2)\{0,-a\} = G \setminus \{a\},$$

which means that $G \setminus \{a\}$ is a fixed point of f. It is easy to verify that $X \in \mathcal{P}_0(G)$ divides $G \setminus \{a\}$ if and only if $a \notin X$. In conclusion, since X divides $G \setminus \{a\}$ if and only if $f(X)$ divides $G \setminus \{a\}$, we obtain that $a \notin X$ if and only if $a \notin f(X)$ and hence $X = f(X)$. □

A counterexample

Example

Let G be isomorphic to C_2^2 (the Klein-four group). The automorphisms of $\mathcal{P}_0(G)$ are precisely the cardinality-prerserving bijections, which means that any automorphism is uniquely determined by a permutation on the 2-element sets and a permutation on the 3-element sets. In conclusion, $\mathrm{Aut}(\mathcal{P}_0(G)) \simeq S_3 \times S_3$, whereas $\mathrm{Aut}(G) \simeq S_3$.

- However, it turns out that this is the only exception.

Theorem (R., 202?)

Let $(G, +)$ be a finite abelian group. Then $\mathrm{Aut}(\mathcal{P}_0(G)) \simeq \mathrm{Aut}(G)$ if and only if $G \not\simeq C_2^2$.

Approaching the proof

- We approach the proof by induction on $|G|$.
- Let G be a finite abelian group, let H be a subgroup of G and f an automorphism of $\mathcal{P}_0(G)$ with trivial pullback.
- Since $f(H) = H$ and $X \subseteq H$ if and only if $X + H = H$, we see that f restricts to an automorphism of $\mathcal{P}_0(H)$.
- Consider the subsemigroup $\mathcal{P}_{0,H}(G) := H + \mathcal{P}_0(G)$ of $\mathcal{P}_0(G)$, which is a monoid with identity H (but not a submonoid of $\mathcal{P}_0(G)$).
- There is a canonical isomorphism $F : \mathcal{P}_{0,H}(G) \to \mathcal{P}_0(G/H)$ via $x + H \subseteq X$ if and only if $x + H \in F(X)$ and we obtain an automorphism $f_{G/H}$ of $\mathcal{P}_0(G/H)$ induced by f.

The induction step

Proposition

Let G be a finite abelian group with $G \not\simeq C_2^2$ and let f be an automorphism of $\mathcal{P}_0(G)$ with trivial pullback. Suppose that the following two conditions hold.

1. For every (maximal) proper subgroup H of G, f restricts to the identity on $\mathcal{P}_0(H)$.
2. For every subgroup H of G of prime order, the induced automorphism $f_{G/H}$ is the identity.

Then f is the identity.

Sketch of proof

Show that $G \setminus \{a\}$ is a fixed point of f for every nonzero $a \in G$. Then show that $a \notin X$ if and only if X divides either $G \setminus \{a\}$ or $H + Y$, where H is a subgroup of G of prime order and $Y \in \mathcal{P}_0(G)$ such that $a \notin H + Y$. Since $f_{G/H}$ is the identity, $H + Y$ is a fixed point of f. This suffices to conclude that $a \notin X$ if and only if $a \notin f(X)$.

Additional base cases

- Since C_2^2 is an exceptional case, we need additional base cases for our induction.
- We need to show the result manually for groups isomorphic to C_2^3 and $C_2 \oplus C_{2p}$, where p is a prime number. These are precisely the finite abelian groups, which contain C_2^2 as a maximal proper subgroup.
- Once this is done, the theorem can be easily proven.
- Can this be generalized to other types of groups?

Submonoids of $(\mathbb{Q}, +)$ (joint work with Weihao Yan)

- Let H be a submonoid of $(\mathbb{Q}, +)$, i.e. either a subgroup of $\mathbb{Q}$ or (up to isomorphism) a *rational Puiseux monoid*, that is, a submonoid of $\mathbb{Q}_{\geq 0}$.
- We call H a *cone* if $H = \mathbb{Q}_{\geq 0} \cap \mathsf{q}(H)$. E.g. $H = \mathbb{N}_0$.

Theorem (R.-Yan, 202?)

Let H be a submonoid of $(\mathbb{Q}, +)$. Then

$$\mathrm{Aut}(\mathcal{P}_{\mathrm{fin},0}(H)) \simeq \begin{cases} \mathrm{Aut}(H) \times C_2, & \text{if } H \text{ is a cone} \\ \mathrm{Aut}(H), & \text{otherwise} \end{cases}$$

- Here, C_2 corresponds to the (generalized) reversion map

$$\mathrm{rev}(X) = \max(X) + \min(X) - X.$$

Sketch of the proof

- A priori, it makes sense that the reversion map only works for cones.
- If $H = \mathbb{Z}$, then $\text{rev}(\{-1, 0, 2\}) = \{-1, 1, 2\}$, which does not contain 0.
- If $H = \langle 2, 3 \rangle$, then $\text{rev}(\{0, 2, 3\}) = \{0, 1, 3\}$. However, $1 \notin H$.
- We illustrate the core idea of the proof in the specific case $H = \mathbb{N}_0$.
- Let $X \in \mathcal{P}_{\text{fin},0}(H)$. We call $d \in \mathbb{N}$ a *difference* in X if there is an ordered pair $(x, y) \in X \times X$ such that $x - y = d$ and we call the number of such ordered pairs the *multiplicity* of d, denoted by $m_X(d)$.
- Again, in this setting the pullback of an automorphism $f : \mathcal{P}_{\text{fin},0}(H) \to \mathcal{P}_{\text{fin},0}(H)$ is itself an automorphism and we will assume that it is the identity.

Sketch of the proof

- Take $d \in \mathbb{N}$, a sufficiently large $n \in \mathbb{N}$ and set

$$X_n = X + \{0, (2n-1)d\}.$$

- The number of sets Y, satisfying $X_n + \{0, nd\} = Y + \{0, nd\}$ is precisely $2^{m_X(d)}$.
- It follows that differences and their multiplicities are preserved under automorphisms of $\mathcal{P}_{\text{fin},0}(H)$. In particular, automorphisms are cardinality-preserving.
- After a couple of similar steps, we obtain that either $f(X) = X$ or $f(X) = \text{rev}(X)$.
- It seems plausible that the approach can be generalized to higher dimensions, i.e. to submonoids of $\mathbb{Q}^n$. However, the pullback can fail to be a homomorphism already for submonoids of $\mathbb{Z}^2$ (R. 202?).

Thank you for your attention!