Power monoids and their system of length sets

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Terminology I

In this talk a monoid is always a commutative semigroup with identity. If not stated otherwise, then monoids are written additively. Let M be a monoid.

Set $M^\times = \{x \in M \mid x + y = 0 \text{ for some } y \in M\}$, called the **unit group** of M.

Set $\mathcal{A}(M) = \{x \in M \setminus M^\times \mid \text{for all } y, z \in M, x = y + z \text{ implies } y \in M^\times \text{ or } z \in M^\times\}$, called the **set of atoms** of M.

Terminology II

For each $a \in M$, let $\mathsf{L}(a) = \{n \in \mathbb{N}_0 \mid a = \sum_{i=1}^{n} u_i$ for some atoms $(u_i)_{i=1}^{n}$ of $M\}$, called the **length set** of a.

Let $\mathcal{L}(M) = \{\mathsf{L}(a) \mid a \in M, \mathsf{L}(a) \neq \emptyset\}$, called the **system of length sets** of M.

We say that M is **nontorsion** if $\{na \mid n \in \mathbb{N}\}$ is infinite for some $a \in M$.

We say that M is **reduced** if $M^{\times} = \{0\}$.

If $a, b \in M$, then we say that a and b are **relatively prime** (**rel. prime** for short) if for all $c, d, e \in M$ with $a = c + d$ and $b = c + e$, it follows that $c \in M^{\times}$.

Terminology III

For each $a \in M$, let

$$\rho(a) = \begin{cases} \sup(\mathsf{L}(a))/\min(\mathsf{L}(a)) & \text{if } \min(\mathsf{L}(a)) > 0 \\ 1 & \text{else} \end{cases},$$

called the **elasticity** of a.

Set $\rho(M) = \sup\{\rho(a) \mid a \in M\}$, called the **elasticity** of M.

We say that M is **atomic** if every nonunit of M is a finite sum of atoms.

Furthermore, M is said to be **fully elastic** if M is atomic and for each rational number $1 \le r < \rho(M)$, there is some $a \in M$ such that $\rho(a) = r$.

Terminology IV

Let M be reduced. Let $\mathsf{Z}(M)$ be the free abelian (multiplicatively written) monoid with basis $\mathcal{A}(M)$, called the **factorization monoid**.

Let $\pi : \mathsf{Z}(M) \to M$ be the **factorization homomorphism** defined by $\pi(\prod_{i=1}^{n} a_i) = \sum_{i=1}^{n} a_i$ for all $n \in \mathbb{N}_0$ and atoms $(a_i)_{i=1}^{n}$ of M.

For each $a \in M$, set $\mathsf{Z}(a) = \{z \in \mathsf{Z}(M) \mid \pi(z) = a\}$, called the **set of factorizations** of a.

Cancellative and relatively cancellative elements

Let $a \in M$. We say that a is **cancellative** if for all $b, c \in M$ with $a + b = a + c$, it follows that $b = c$.
Moreover, a is called **relatively cancellative** if for all $b, c, d \in M$ with $a = b + c = b + d$, it follows that $c = d$.

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Observe that units are cancellative and cancellative elements are relatively cancellative. If M is reduced, then every atom is relatively cancellative (but not necessarily cancellative). It is also straightforward to prove that divisors of (relatively) cancellative elements are (relatively) cancellative. Sums of cancellative elements are cancellative, but sums of relatively cancellative elements need not be relatively cancellative.

Power monoids

Let $\mathcal{P}_{\text{fin}}(M)$ be the set of nonempty finite subsets of M and let $\mathcal{P}_{\text{fin},0}(M)$ be the set of finite subsets of M that contain 0.

Let $+ : \mathcal{P}_{\text{fin}}(M) \times \mathcal{P}_{\text{fin}}(M) \to \mathcal{P}_{\text{fin}}(M)$ be defined by $A + B = \{a + b \mid a \in A, b \in B\}$.

Then $\mathcal{P}_{\text{fin}}(M)$ equipped with $+$ is a monoid, called the **finitary power monoid** of M.

Besides that $\mathcal{P}_{\text{fin},0}(M)$ is a submonoid of $\mathcal{P}_{\text{fin}}(M)$, called the **reduced finitary power monoid** of M.

Some motivational results I

Theorem (Chapman, McClain, 2005)

Let $D = \{f \in \mathbb{Q}[X] \mid f(\mathbb{Z}) \subseteq \mathbb{Z}\}$ be the ring of integer-valued polynomials and let $H = D \setminus \{0\}$.
Then H is fully elastic.

Some motivational results I

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Theorem (Frisch, 2013)

Let $D = \{f \in \mathbb{Q}[X] \mid f(\mathbb{Z}) \subseteq \mathbb{Z}\}$ be the ring of integer-valued polynomials and let $H = D \setminus \{0\}$.
Then $\mathcal{L}(H) = \{\{0\}, \{1\}\} \cup \{A \subseteq \mathbb{N}_0 \mid \emptyset \neq A \text{ is finite}, \min(A) \geq 2\}$.

Some motivational results II

Theorem (Kainrath, 1999)

Let H be a Krull monoid with quotient group K and let G be the divisor class group of H. Suppose that G is infinite and for every $\mathcal{C} \in G$, there is some height-one prime ideal P of H such that $\mathcal{C} = \{cP \mid c \in K\}$.
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Theorem (Geroldinger, Kainrath, 202?)

Let G be an infinite abelian group and let $H = \mathcal{B}_{\pm}(G)$ be the monoid of plus-minus weighted zero-sum sequences.
Then $\mathcal{L}(H) = \{\{0\}, \{1\}\} \cup \{A \subseteq \mathbb{N}_0 \mid \emptyset \neq A \text{ is finite, } \min(A) \geq 2\}$.

A conjecture and some partial results

Conjecture (Fan, Tringali, 2018)

$\mathcal{L}(\mathcal{P}_{\mathrm{fin},0}(\mathbb{N}_0)) = \{\{0\},\{1\}\} \cup \{A \subseteq \mathbb{N}_0 \mid \emptyset \neq A \text{ finite}, \min(A) \geq 2\}$.

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Theorem (Fan, Tringali, 2018)

$\{[2,n],\{2,n\},\{n\} \mid n \in \mathbb{N}_{\geq 2}\} \subseteq \mathcal{L}(\mathcal{P}_{\mathrm{fin},0}(\mathbb{N}_0))$.

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Corollary

$\mathcal{P}_{\mathrm{fin}}(\mathbb{N}_0)$ is fully elastic.

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Corollary

$\mathcal{P}_{\mathrm{fin}}(\mathbb{N}_0)$ is fully elastic.

Theorem (Antoniou, 2020)

$\{[2,n] \cup \{n+m\} \mid n,m \in \mathbb{N}_{\geq 2}\} \subseteq \mathcal{L}(\mathcal{P}_{\mathrm{fin},0}(\mathbb{N}_0))$.

Why relatively cancellative elements are useful

Proposition (R., 2025)

Let $\mathcal{H} = \mathcal{P}_{\mathrm{fin},0}(\mathbb{N}_0)$ and let $X, Y \in \mathcal{H}$ be relatively cancellative such that $\gcd(Y) > 2\max(X)$.
Then $X + Y$ is relatively cancellative, $\mathsf{Z}(X + Y) = \mathsf{Z}(X)\mathsf{Z}(Y)$ and $\mathsf{L}(X + Y) = \mathsf{L}(X) + \mathsf{L}(Y)$.

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Corollary

Let $\mathcal{H} = \mathcal{P}_{\mathrm{fin},0}(\mathbb{N}_0)$ and let $X, Y \in \mathcal{H}$ be relatively cancellative.
Then there is some relatively cancellative $Z \in \mathcal{H}$ such that $\mathsf{L}(X) + \mathsf{L}(Y) = \mathsf{L}(Z)$.

Some counterexamples

Examples

Let $A = \{0, 1, 2, 3\}$, $B = \{0, 7\}$, $C = \{0, 1\}$, $D = \{0, 3, 6, 9\}$, $E = \{0, 2\}$.

(1) A and D are not relatively cancellative and B, C and E are relatively cancellative.

(2) $\gcd(B) > 2\max(A)$, $\gcd(D) > 2\max(C)$ and $\gcd(E) = 2\max(C)$.

(3) $\mathsf{L}(A + B) = \{2, 3, 4\} \supsetneq \{3, 4\} = \mathsf{L}(A) + \mathsf{L}(B)$.

(4) $\mathsf{L}(C + D) = \{2, 3, 4\} \supsetneq \{3, 4\} = \mathsf{L}(C) + \mathsf{L}(D)$.

(5) $\mathsf{L}(C + E) = \{2, 3\} \supsetneq \{2\} = \mathsf{L}(C) + \mathsf{L}(E)$.

How to construct nontrivial relatively cancellative elements

Theorem (R., 2025)

Let $\mathcal{H} = \mathcal{P}_{\text{fin},0}(\mathbb{N}_0)$ and let $(n_j)_{j=0}^{\infty}$, $(A_j)_{j=0}^{\infty}$, $(B_j)_{j=0}^{\infty}$, $(C_j)_{j=0}^{\infty}$, $(D_j)_{j=0}^{\infty}$ and $(S_j)_{j=0}^{\infty}$ be defined recursively as follows.
Set $n_0 = 0$, $A_0 = \{0,1\}$, $B_0 = \{0\}$,
$C_0 = \{0\}$, $D_0 = \{0,1\}$ and $S_0 = \{0,1\}$.
For each $i \in \mathbb{N}_0$, let $n_{i+1} \in \mathbb{N}$ with $n_{i+1} \geq 3\max(D_i)$,
$A_{i+1} = A_i \cup \{1 + n_{i+1}\}$, $B_{i+1} = B_i \cup (n_{i+1} + D_i)$,
$C_{i+1} = C_i \cup \{n_{i+1}\}$, $D_{i+1} = D_i + \{0, 1 + n_{i+1}\}$ and
$S_{i+1} = C_{i+1} + D_{i+1}$.
Then for each $i \in \mathbb{N}$, $S_i \in \mathcal{H}$ is relatively cancellative,
$\mathsf{Z}(S_i) = \{A_i B_i, C_i \prod_{j=0}^{i}\{0, 1 + n_j\}\}$ and $\mathsf{L}(S_i) = \{2, i+2\}$.

Sketch of the proof I

Step 1: Show by induction that for each $i \in \mathbb{N}$, $A_i, B_i, C_i \in \mathcal{A}(\mathcal{H})$, $D_i \in \mathcal{H}$, $\mathsf{Z}(D_i) = \{\prod_{j=0}^{i}\{0, 1 + n_j\}\}$, $\mathsf{L}(D_i) = \{i+1\}$, $\max(A_i \cup B_i \cup C_i) < \max(D_i)$, $A_i = \{0\} \cup (1 + C_i)$ and $A_i + B_i = C_i + D_i$.

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Step 2: Prove that for each $i \in \mathbb{N}$, $\mathsf{Z}(S_i) = \{A_i B_i, C_i \prod_{j=0}^{i}\{0, 1+n_j\}\}$.

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Step 2: Prove that for each $i \in \mathbb{N}$, $\mathsf{Z}(S_i) = \{A_i B_i, C_i \prod_{j=0}^{i}\{0, 1+n_j\}\}$.

Step 2a: Show that $\mathsf{Z}(S_1) = \{A_1 B_1, C_1 \prod_{j=0}^{1}\{0, 1+n_j\}\}$ (straightforward) and observe that $\{A_i B_i, C_i \prod_{j=0}^{i}\{0, 1+n_j\}\} \subseteq \mathsf{Z}(S_i)$ for each $i \in \mathbb{N}$ by Step 1.

Sketch of the proof II

Step 2b: Let $i \in \mathbb{N}$ be such that $\mathsf{Z}(S_i) \subseteq \{A_i B_i, C_i \prod_{j=0}^{i} \{0, 1 + n_j\}\}$. Prove that $\mathsf{Z}(S_{i+1}) \subseteq \{A_{i+1} B_{i+1}, C_{i+1} \prod_{j=0}^{i+1} \{0, 1 + n_j\}\}$.

Sketch of the proof II

Step 2b: Let $i \in \mathbb{N}$ be such that $\mathsf{Z}(S_i) \subseteq \{A_i B_i, C_i \prod_{j=0}^{i} \{0, 1+n_j\}\}$. Prove that $\mathsf{Z}(S_{i+1}) \subseteq \{A_{i+1} B_{i+1}, C_{i+1} \prod_{j=0}^{i+1} \{0, 1+n_j\}\}$.

Step 2b(i): Let $z \in \mathsf{Z}(S_{i+1})$. Then $z = \prod_{i=1}^{n+1} Y_i$ with $n \in \mathbb{N}$ and $(Y_i)_{i=1}^{n+1}$ atoms of $\mathcal{H}$. There is some $j \in [1, n+1]$ such that $n_1 \in Y_j$. Without restriction, let $n_1 \in Y_{n+1}$. Set $X = \sum_{i=1}^{n} Y_i$ and $Y = Y_{n+1}$. Then $S_{i+1} = X + Y$, $C_{i+1} \subseteq Y$ and $X \subseteq D_{i+1}$.

Sketch of the proof II

Step 2b: Let $i \in \mathbb{N}$ be such that
$\mathsf{Z}(S_i) \subseteq \{A_iB_i, C_i \prod_{j=0}^{i}\{0, 1+n_j\}\}$. Prove that
$\mathsf{Z}(S_{i+1}) \subseteq \{A_{i+1}B_{i+1}, C_{i+1} \prod_{j=0}^{i+1}\{0, 1+n_j\}\}$.

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Set $X' = X \cap [0, n_{i+1} - 1]$ and $Y' = Y \cap [0, n_{i+1} - 1]$. Show that $S_i = X' + Y'$.

Conclude that either (1) $X' = A_i$ and $Y' = B_i$ or (2) there is some $W \subseteq [0, i]$ such that $X' = \{\sum_{j \in E}(1 + n_j) \mid E \subseteq W\}$ and $Y' = \{\sum_{j \in E}(1 + n_j) \mid E \subseteq [0, i] \setminus W\} + C_i$.

Sketch of the proof III

Step 2b(ii): First let $X' = A_i$ and $Y' = B_i$.
Prove that $X = A_{i+1}$ and $Y = B_{i+1}$.
Then $z = A_{i+1}B_{i+1}$.

Sketch of the proof III

Step 2b(ii): First let $X' = A_i$ and $Y' = B_i$.
Prove that $X = A_{i+1}$ and $Y = B_{i+1}$.
Then $z = A_{i+1}B_{i+1}$.

Next let $W \subseteq [0, i]$ be such that $X' = \{\sum_{j \in E}(1 + n_j) \mid E \subseteq W\}$ and $Y' = \{\sum_{j \in E}(1 + n_j) \mid E \subseteq [0, i] \setminus W\} + C_i$.
Show that $W = [0, i]$, $X = \{\sum_{j \in E}(1 + n_j) \mid E \subseteq [0, i + 1]\}$ and $Y = C_{i+1}$.
Then $z = C_{i+1} \prod_{j=0}^{i+1}\{0, 1 + n_j\}$.

Sketch of the proof III

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Prove that $X = A_{i+1}$ and $Y = B_{i+1}$.
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Next let $W \subseteq [0, i]$ be such that $X' = \{\sum_{j \in E}(1 + n_j) \mid E \subseteq W\}$ and $Y' = \{\sum_{j \in E}(1 + n_j) \mid E \subseteq [0, i] \setminus W\} + C_i$.
Show that $W = [0, i]$, $X = \{\sum_{j \in E}(1 + n_j) \mid E \subseteq [0, i + 1]\}$ and $Y = C_{i+1}$.
Then $z = C_{i+1} \prod_{j=0}^{i+1}\{0, 1 + n_j\}$.

Step 3: Let $i \in \mathbb{N}$. Conclude that $S_i \in \mathcal{H}$ is relatively cancellative and $\mathsf{L}(S_i) = \{2, i + 2\}$. (The former follows from the fact that for all $u, v \in \mathsf{L}(S_i)$ with $\gcd(u, v) \neq 1$, it follows that $u = v$ and the latter is obvious.)

A few consequences

Corollary

Let $\mathcal{U} \subseteq \mathcal{P}_{\mathrm{fin}}(\mathbb{N}_0)$ be the submonoid generated by $\{\{1\}, \{2, n\} \mid n \in \mathbb{N}_{\geq 3}\}$. Then $\mathcal{U} \subseteq \mathcal{L}(\mathcal{P}_{\mathrm{fin},0}(\mathbb{N}_0))$.

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Let $\mathcal{U} \subseteq \mathcal{P}_{\text{fin}}(\mathbb{N}_0)$ be the submonoid generated by $\{\{1\}, \{2, n\} \mid n \in \mathbb{N}_{\geq 3}\}$. Then $\mathcal{U} \subseteq \mathcal{L}(\mathcal{P}_{\text{fin},0}(\mathbb{N}_0))$.

Corollary

If $L \subseteq \mathbb{N}_{\geq 2}$ is a nonempty finite arithmetical progression of length n and $\min(L) \geq 2n$, then $L \in \mathcal{L}(\mathcal{P}_{\text{fin},0}(\mathbb{N}_0))$.

A complementary result

Proposition (R., 2025)

Let $\mathcal{H} = \mathcal{P}_{\mathrm{fin},0}(\mathbb{N}_0)$ and let $X \in \mathcal{H}$ and $n \in \mathbb{N}$ be such that $n > 2\max(X)$.
Set $\mathcal{M} = \{(A, B, C) \in \mathcal{H}^3 \mid A + B = A + C = X, B$ and C are rel. prime$\}$.
Set $\mathcal{N} = \{A \in \mathcal{H} \mid$ there are $B, C \in \mathcal{H}$ such that $(A, B, C) \in \mathcal{M}\}$.
Then $\mathsf{Z}(X + \{0, n\}) = \bigcup_{(A,B,C)\in\mathcal{M}} (B \cup (n + C))\mathsf{Z}(A)$ and $\mathsf{L}(X + \{0, n\}) = 1 + \bigcup_{A\in\mathcal{N}} \mathsf{L}(A)$.

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Corollary

$[k, k + 2] \in \mathcal{L}(\mathcal{P}_{\mathrm{fin},0}(\mathbb{N}_0))$ for each $k \in \mathbb{N}_{\geq 2}$.

Examples I

Example

Let $X = \{0, 1, 2, 4, 5, 6, 10, 11, 12, 14, 15, 16, 20, 21, 22, 24, 25, 26\}$.

$$
\begin{aligned}
X =& \{0, 1\} + \{0, 1\} + \{0, 4\} + \{0, 10\} + \{0, 10\} \\
=& \{0, 4\} + \{0, 10\} + \{0, 1, 2, 6, 10, 11, 12\} \\
=& \{0, 4\} + \{0, 1, 2, 6, 10, 11, 12, 20, 21, 22\}.
\end{aligned}
$$

Besides that, X is not relatively cancellative and $\mathsf{L}(X) = \{2, 3, 5\}$.

Examples II

Example

$X = \{0,1,3,4,5,7,8,13,14,16,17,18,20,21,22,24,25,30\} \cup \{31,33,34,35,37,38\}$.

$$\begin{aligned} X =& \{0,1\} + \{0,3\} + \{0,4\} + \{0,13\} + \{0,17\} \\ =& \{0,1,4\} + \{0,3,4\} + \{0,13\} + \{0,17\} \\ =& \{0,4\} + \{0,1,3,4,13,14,16,18,20,21,30,31,33,34\}. \end{aligned}$$

Furthermore, X is not relatively cancellative and $\mathsf{L}(X) = \{2,4,5\}$.

Examples III

Example

Let $X = \{0, 1, 4, 5, 10, 11, 12, 14, 15, 16, 19, 20, 21, 22, 25, 26, 29\} \cup \{30, 42, 43, 46, 47, 52, 53, 54, 56, 57, 58, 61, 62, 63, 64, 67, 68, 71, 72\}$.

$$\begin{aligned} X =& \{0,1\} + \{0,4\} + \{0,10\} + \{0,11,15\} + \{0,42\} \\ =& \{0,1\} + \{0,10\} + \{0,4,10,11,15,19\} + \{0,42\} \\ =& \{0,10,11\} + \{0,1,4,5,10,11,15,19\} + \{0,42\}. \end{aligned}$$

Moreover, X is not relatively cancellative and $\mathsf{L}(X) = \{3, 4, 5\}$.

Another application

Theorem (R., 2025)

Let G be an abelian group, let M be an atomic nontorsion submonoid of G and let $\mathcal{H} \in \{\mathcal{P}_{\mathrm{fin}}(M), \mathcal{P}_{\mathrm{fin},0}(M)\}$.

(1) The submonoid of $\mathcal{P}_{\mathrm{fin}}(\mathbb{N}_0)$ generated by $\{\{1\}, \{2, n\} \mid n \in \mathbb{N}_{\geq 3}\}$ is contained in $\mathcal{L}(\mathcal{H})$.

(2) $\{[k, k+2] \mid k \in \mathbb{N}_{\geq 2}\} \subseteq \mathcal{L}(\mathcal{H})$.

(3) $\mathcal{H}$ is fully elastic.

Open problems

Let $\mathcal{H} = \mathcal{P}_{\text{fin},0}(\mathbb{N}_0)$.

(1) Is $\mathcal{L}(\mathcal{H}) = \{\mathsf{L}(A) \mid A \in \mathcal{H} \text{ is relatively cancellative}\}$? In particular, is there some relatively cancellative $X \in \mathcal{H}$ such that $\mathsf{L}(X) = \{2, 3, 4\}$ or $\mathsf{L}(X) = \{3, 4, 5\}$?

(2) Are all nonempty finite arithmetical progressions $L \subseteq \mathbb{N}_{\geq 2}$ length sets of $\mathcal{H}$?

(3) Is the Conjecture of Fan and Tringali true?

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Thank you for your attention!